

The least error method for sparse solution reconstruction

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Abstract. This work deals with a regularization method enforcing solution sparsity of linear ill-posed problems by appropriate discretization in the image space. Namely, we formulate the so called least error method in an ℓ^1 setting and perform the convergence analysis by choosing the discretization level according to an a priori rule, as well as two a posteriori rules, via the discrepancy principle and the monotone error rule, respectively. Depending on the setting, linear or sublinear convergence rates in the ℓ^1 -norm are obtained under a source condition yielding sparsity of the solution. A part of the study is devoted to analyzing the structure of the approximate solutions and of the involved source elements.

1. Introduction

In order to recover sparse solutions of linear operator equations, it is common to consider Tikhonov regularization with ℓ^1 -penalty (see, e.g. [4]). In this paper, we focus on a different regularization method based on discretization known in the literature as the least error or the dual least squares method, and taking advantage of the ℓ^1 framework. The reader is referred e.g., to [9], [10], [5] for some classical analysis of the above method in Hilbert spaces and to [8], for recent results in some classes of Banach spaces. Thus, it has been shown in [8] that the least error method converges in spaces with good smoothness and convexity properties, which is not the case in the considered sparsity context. Thus, to the best knowledge of the authors, this is the first time the least error approach is analyzed in the context of sparse regularization in ℓ^1 . Technically speaking, this analysis differs essentially as regards stability estimates and convergence of the method for an a priori rule, which is the backbone of convergence for the method combined with the monotone error rule or the discrepancy principle for choosing the discretization level, playing the role of a regularization parameter here. Under a source condition we get a convergence rate result, not only for the Bregman distance, but even for the full ℓ^1 norm. This is a consequence of the sparsity structure of the exact solution induced by the source condition, that enables a special stability estimate and an ideal error rate $O(\delta)$ as the noise level δ tends to zero, under certain a priori information, similar to [7] and [2, 6] for the case of non-convex sparse regularization. A convergence rate with a posteriori choice of the discretization level can be alternatively obtained with the discrepancy principle.

Throughout the paper, let H be a Hilbert space and $A : H \rightarrow c_0$ be linear and continuous. Then, with the identification $c_0^* = \ell^1$, the mapping $A^* : \ell^1 \rightarrow H$ is weak*-to-weak continuous in addition to being linear and continuous. For the time being, we do not assume that A is injective, but will make this assumption later, observing already that A^* will also be injective in this case.

We would like to solve the inverse problem

$$A^*u = f \tag{1}$$

provided only data f^δ satisfying

$$\|f^\delta - f\|_H \leq \delta, \tag{2}$$

an assumption that is also made throughout the paper.

The aim of this study is to solve the equation by discretization in the image space H . That is, choose a sequence of subspaces $(H_n)_n$ of H where each H_n is finite-dimensional with dimension n and $\lim_{n \rightarrow \infty} P_n v = v$ for each $v \in H$, with P_n denoting the orthogonal projection onto H_n , i.e.,

$$P_n = \text{Proj}_{H_n}. \tag{3}$$

The least error method defines

$$u^n \in \arg \min \{ \|u\|_1 : \forall z^n \in H_n : \langle z^n, A^*u \rangle = \langle z^n, f^\delta \rangle \}, \tag{4}$$

which is equivalent to u^n solving

$$\min_{u \in \ell^1} \|u\|_1 \quad \text{subject to} \quad P_n A^* u = P_n f^\delta. \quad (5)$$

The structure of this work is as follows. Well-definedness, an equivalent formulation of the least error method and a few useful estimates are shown in Section 2. A convergence analysis for an a priori choice of the discretization level, as well as for two a posteriori choices is provided in Section 3, 4, respectively. Convergence rates up to $O(\delta)$ are derived in Section 5, where the specific structure of the (approximate) solutions and the corresponding source elements are also discussed. Section 6 shortly reviews some particular instances of the least error method in the current setting.

2. The least error method in ℓ^1

We will use the identification of ℓ^1 with the dual of the space c_0 of sequences converging to zero and the weak* compactness of the sublevel sets of the ℓ^1 norm. Recall that

$$\partial(\|\cdot\|_1)(u) = \{\xi \in \ell^\infty : \|\xi\|_\infty \leq 1 \quad \text{and} \quad \langle \xi, u \rangle = \|u\|_1\} \quad (6)$$

due to convexity and homogeneity of the ℓ^1 -norm. More specifically, by exploiting the structure of the ℓ^1 -norm, we have

$$\partial(\|\cdot\|_1)(u) = \text{sgn}(u) = \{\xi \in \ell^\infty \mid \xi_i = \frac{u_i}{|u_i|} \text{ if } u_i \neq 0, \xi_i \in [-1, 1] \text{ if } u_i = 0\}. \quad (7)$$

Problem (4) is well-defined, as stated below.

Proposition 1 Assume that

$$\mathcal{N}(A) \cap H_n = \{0\}. \quad (8)$$

Then the set of minimizers $\arg \min\{\|u\|_1 : \forall z^n \in H_n : \langle z^n, A^* u \rangle = \langle z^n, f^\delta \rangle\}$ is nonempty.

Proof: The proof is similar to the one in [8], showing that the feasible set is nonempty. Since (8) implies $\mathcal{N}(AP_n) = \{0\}$ and the range of the operator $P_n A^*$ is finite dimensional, hence closed, we can conclude

$$\mathcal{R}(P_n A^*) = \mathcal{R}((AP_n)^*) = \mathcal{N}(AP_n)^\perp = H_n.$$

Weak*-weak sequential continuity (which is ensured here, as mentioned above) of the operator A^* implies weak*-closedness of the feasible set. This together with coercivity of the objective function and weak* compactness of the sublevel sets of the ℓ^1 norm yield the result. \square

Note that weak*-weak sequential continuity of the operator governing the equation to be solved has been considered in [3], in the context of Tikhonov type regularization.

Proposition 2 Let the assumptions of Proposition 1 be satisfied. Then (4) is equivalent to

$$u^n \in E_n \text{ and } \forall z^n \in H_n : \langle z^n, A^*u^n \rangle = \langle z^n, f^\delta \rangle,$$

where

$$E_n = (\partial \|\cdot\|_1)^{-1}(AH_n). \quad (9)$$

Proof: Let $H_n = \text{span}\{e_1, \dots, e_n\}$, i.e., the elements e_i form a basis of H_n . Then problem (4) is equivalent to

$$u^n \in \text{argmin}\{\|u\|_1 : G(u) = 0\}, \quad (10)$$

where $G : \ell^1 \rightarrow \mathbb{R}^n$ is given by $G(u) = Tu + b$ with $Tu = (\langle A^*u, e_i \rangle)_i$ and $b = (-\langle f^\delta, e_i \rangle)_i$. Since the function $\phi = \|\cdot\|_1$ is continuous and the finite dimensional rank operator T has a closed range, one can apply Th. 3.20 in [1] and obtain that u^n is a solution of (10) if and only if

$$\partial\phi(u^n) \cap \mathcal{R}(T^*) \neq \emptyset.$$

As for any $u \in \ell^1$, $p \in \mathbb{R}^n$

$$(Tu)^T p = \sum_{i=1}^n \langle A^*u, e_i \rangle p_i = \langle u, A \sum_{i=1}^n p_i e_i \rangle$$

and therefore $\mathcal{R}(T^*) = A(\text{span}\{e_1, \dots, e_n\}) = AH_n$, the proof is complete. \square

Remark 1 For interpreting the optimality conditions derived in Proposition 2, we recall that

$$(\partial \|\cdot\|_1)^{-1} = \partial(I_{\{\|\xi\|_\infty \leq 1\}}),$$

where the subgradient has to be understood as subset of the predual space ℓ^1 and reads as

$$\begin{aligned} \|\xi\|_\infty > 1 : \quad & \partial(I_{\{\|\xi\|_\infty \leq 1\}})(\xi) = \emptyset, \\ \|\xi\|_\infty \leq 1 : \quad & \partial(I_{\{\|\xi\|_\infty \leq 1\}})(\xi) = \{u \in \ell^1 \mid u_i = 0 \text{ if } \xi_i \in]-1, 1[, \\ & u_i \geq 0 \text{ if } \xi_i = 1 \text{ and } u_i \leq 0 \text{ if } \xi_i = -1\}. \end{aligned}$$

Thus, according to Proposition 2, u^n solves (4) iff there exists an element $v^n \in H_n$ such that

$$\|Av^n\|_\infty \leq 1 \text{ and } (u^n)_i \begin{cases} = 0 & \text{if } (Av^n)_i \in]-1, 1[\\ \geq 0 & \text{if } (Av^n)_i = 1 \\ \leq 0 & \text{if } (Av^n)_i = -1 \end{cases}$$

In order to show stability of the discretization method, we define

$$\kappa_n = \sup_{z^n \in H_n} \frac{\|z^n\|}{\|Az^n\|_\infty} = \frac{1}{\inf_{\|z^n\|=1} \|Az^n\|_\infty}. \quad (11)$$

Note that these values are finite due to (8).

The Bregman distance with respect to the ℓ^1 norm and an element $\xi_v \in \partial \|\cdot\|_1(v)$ is defined by

$$D(u, v) = \|u\|_1 - \|v\|_1 - \langle \xi_v, u - v \rangle = \|u\|_1 - \langle \xi_v, u \rangle,$$

see (6), the symmetric Bregman distance by

$$D^{\text{sym}}(v, u) = D(v, u) + D(u, v) = \langle \xi_v - \xi_u, v - u \rangle, \quad (12)$$

with $\xi_u \in \partial \|\cdot\|_1(u)$.

Lemma 1 (compare [8, Lemma 4.5]) Under the conditions of Proposition 1,

$$\|u^n\|_1 \leq \delta \kappa_n + \|u^\dagger\|_1 \quad (13)$$

holds.

Proof: Due to Proposition 2, each solution u^n of (4) satisfies $\xi^n = Av^n \in \partial \|\cdot\|_1(u^n)$ for some $v^n \in H_n$. Thus,

$$\begin{aligned} \|u^n\|_1 &= \langle \xi^n, u^n \rangle = \langle Av^n, u^n \rangle \\ &= \langle v^n, A^* u^n \rangle = \langle v^n, f^\delta \rangle \\ &= \langle v^n, f^\delta - f \rangle + \langle Av^n, u^\dagger \rangle \\ &\leq \delta \kappa_n \|Av^n\|_\infty + \|u^\dagger\|_1 \|Av^n\|_\infty \\ &\leq \delta \kappa_n + \|u^\dagger\|_1 \end{aligned} \quad \square$$

Proposition 3 Let the assumptions of Proposition 1 be satisfied and let

$$u^{n,i} \in \operatorname{argmin}\{\|u\|_1 : \forall z^n \in H_n : \langle z^n, A^* u \rangle = \langle z^n, f^i \rangle\} \quad \text{for } i = 1, 2,$$

be well-defined solutions of (5) corresponding to data $f^1, f^2 \in H$, respectively.

Then the following estimates hold:

$$D^{\text{sym}}(u^{n,1}, u^{n,2}) \leq 2\kappa_n \|f^1 - f^2\|, \quad (14)$$

$$|\|u^{n,1}\|_1 - \|u^{n,2}\|_1| \leq 2\kappa_n \|f^1 - f^2\|. \quad (15)$$

Proof: Let $v^{n,i} \in H_n$ such that $\xi^{n,i} = Av^{n,i} \in \partial(\|\cdot\|_1)(u^{n,i})$, for $i = 1, 2$. Then (14) follows from

$$\begin{aligned} D^{\text{sym}}(u^{n,1}, u^{n,2}) &= \langle Av^{n,1} - Av^{n,2}, u^{n,1} - u^{n,2} \rangle \\ &= \langle v^{n,1} - v^{n,2}, A^* u^{n,1} - A^* u^{n,2} \rangle \\ &= \langle v^{n,1} - v^{n,2}, f^1 - f^2 \rangle \\ &\leq \kappa_n \|Av^{n,1} - Av^{n,2}\|_\infty \|f^1 - f^2\| \\ &\leq 2\kappa_n \|f^1 - f^2\|, \end{aligned}$$

where the last inequality is a consequence of (6).

In order to show (15), let $u \in \ell^1$ be a solution of (4) with $f^1 - f^2$ instead of f^δ . According to (14), plugging in u and 0 instead of $u^{n,1}$ and $u^{n,2}$ as well as $f^1 - f^2$ and 0 instead of f^1 and f^2 , one has

$$\|u\|_1 = D^{\text{sym}}(u, 0) \leq 2\kappa_n \|f^1 - f^2\|.$$

Since $u^{n,1} + u$ satisfies $\langle z^n, A^*(u^{n,1} + u) \rangle = \langle z^n, f^2 \rangle$, it follows that

$$\|u^{n,2}\|_1 \leq \|u^{n,1} + u\|_1 \leq \|u^{n,1}\|_1 + \|u\|_1 \leq \|u^{n,1}\|_1 + 2\kappa_n \|f^1 - f^2\|,$$

which, by symmetry, implies (15). \square

3. Convergence with a priori choice of n

We state below a convergence result in case of an a priori choice of the discretization dimension n . We will use the following notation for solutions in the exact data case:

$$u^{\dagger,n} \in \operatorname{argmin}\{\|u\|_1 : \forall z^n \in H_n : \langle z^n, A^*u \rangle = \langle z^n, f \rangle\}. \quad (16)$$

Note that for the following convergence result, instead of pointwise convergence of the projections P_n , only a weaker condition is needed for proving convergence with a priori choice of n .

Theorem 1 Let the assumptions of Proposition 1 be satisfied and assume that (1) is solvable. Additionally, assume that

$$\forall z \in H : \inf_{z^n \in H_n} \|A(z - z^n)\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (17)$$

Then the following statements hold:

- (a) For exact data $\delta = 0$ one has convergence

$$\|u^{\dagger,l} - u^\dagger\|_1 \rightarrow 0 \text{ as } l \rightarrow \infty,$$

where $(u^{\dagger,l})_l$ is a subsequence of $(u^{\dagger,n})_n$ with terms given by (16) and u^\dagger is a solution of (1).

- (b) Let the noisy data f^δ satisfy (2), the dimension $n = n_{AP}(\delta)$ be chosen such that

$$n_{AP}(\delta) \rightarrow \infty \text{ and } \delta\kappa_{n_{AP}(\delta)} \rightarrow 0 \text{ as } \delta \rightarrow 0 \quad (18)$$

and the sequence $(\delta_m)_m$ in $(0, +\infty)$ converge to zero. Then there exists a subsequence $(\delta_l)_l$ such that

$$\lim_{l \rightarrow \infty} \|u^l - u^\dagger\|_1 = 0, \quad (19)$$

with $u^l := u^{n_{AP}(\delta_l)}$ and u^\dagger a solution of (1).

Remark 2

Proof: (a) Let u^\dagger be a solution of (1). Due to (16), one has

$$\|u^{\dagger,n}\|_1 \leq \|u^\dagger\|_1. \quad (20)$$

Hence, the sequence $(u^{\dagger,n})_n$ has a weakly* convergent subsequence $(u^{\dagger,l})_l$ with limit point \tilde{u} . Weak*-weak continuity of the operator A^* guarantees weak convergence of $(A^*u^{\dagger,n})_l$ to $A^*\tilde{u}$. Moreover, equality $\langle z^l, A^*u^{\dagger,l} - f \rangle = 0$ for all $z^l \in H_l$ and (17) imply

$$\begin{aligned} \forall z \in H : \langle z, A^*u^{\dagger,l} - f \rangle &= \inf_{z^l \in H_l} \langle z - z^l, A^*u^{\dagger,l} - f \rangle \\ &= \inf_{z^l \in H_l} \langle z - z^l, A^*(u^{\dagger,l} - u^\dagger) \rangle \\ &\leq 2 \inf_{z^l \in H_l} \|A(z - z^l)\| \|u^\dagger\| \rightarrow 0 \text{ as } l \rightarrow \infty. \end{aligned}$$

Consequently, $(A^*u^{\dagger,l})_l$ converges weakly also to f , which means that f must equal $A^*\tilde{u}$. Now (16) and weak* lower semicontinuity of the ℓ^1 norm imply, together with (20),

$$\|\tilde{u}\|_1 \leq \liminf_{l \rightarrow \infty} \|u^{\dagger,l}\|_1 \leq \limsup_{l \rightarrow \infty} \|u^{\dagger,l}\|_1 \leq \|u^\dagger\|_1,$$

that is $\lim_{l \rightarrow \infty} \|u^{\dagger,l}\|_1 = \|\tilde{u}\|_1$. From this and weak* convergence of $(u^{\dagger,l})_l$ to \tilde{u} one deduces

$$\lim_{l \rightarrow \infty} \|u^{\dagger,l} - \tilde{u}\| = 0, \quad (21)$$

based on the Kadec-Klee property in ℓ^1 (see, e.g. [3]).

(b) Denote $n_m := n_{AP}(\delta_m)$ and let u^m be a solution of (4) corresponding to the subspace H_{n_m} and to the noisy data f^{δ_m} . Due to (13), (15), and (18) one obtains boundedness of the sequence $(u^m)_m$. By using the proof idea of a), existence of a subsequence $(u^l)_l$ follows, such that its strong limit point \tilde{u} is a solution of (1). \square

4. Convergence with a posteriori choice of n

Convergence with respect to the a posteriori monotone error rule follows in a manner similar to the one for ‘nice’ spaces — see [8], with some differences due to the space setting. For the sake of completeness, we formulate and prove the result below.

Theorem 2 Let the assumptions of Theorem 1 be satisfied and let u^\dagger be a solution of (1). Then one has

- (a) There exists $v^n \in H_n$ such that $u^n \in (\partial \|\cdot\|_1)^{-1}(Av^n)$.
- (b) The identity $\|u^n\|_1 = \langle v^n, f^\delta \rangle$ holds, where v^n is chosen as in (a). If

$$H_n \subseteq H_{n+1}, \quad (22)$$

then

$$\|u^n\|_1 \leq \|u^{n+1}\|_1.$$

(c) Let $d_{ME}(n)$ stand for

$$d_{ME}(n) = \frac{\langle v^{n+1} - v^n, f^\delta \rangle}{\|v^{n+1} - v^n\|} \quad \text{for } v^{n+1} \neq v^n \quad \text{and} \quad d_{ME}(n) = 0 \quad \text{else,}$$

then the following hold:

$$D(u^\dagger, u^{n+1}) - D(u^\dagger, u^n) \leq -(d_{ME}(n) - \delta) \|v^{n+1} - v^n\|.$$

and, in case $v^{n+1} \neq v^n$,

$$d_{ME}(n) = \frac{\|u^{n+1}\|_1 - \|u^n\|_1}{\|v^{n+1} - v^n\|}$$

where the Bregman distances are with respect to $\xi^{n+1} = Av^{n+1}$ and $\xi^n = Av^n$, respectively. Additionally, if (22) holds, then

$$d_{ME}(n) = \frac{D(u^{n+1}, u^n)}{\|v^{n+1} - v^n\|} \geq 0,$$

and the error measured in the Bregman distance is monotonically decreasing as long as

$$\delta \leq d_{ME}(n). \quad (23)$$

(d) Let (22) hold for all $n \in \mathbb{N}$ and let $n = n_{ME}(\delta)$ be the first index such that (23) is violated

$$n_{ME}(\delta) = \min\{n \in \mathbb{N} : v^{n+1} \neq v^n \text{ and } \frac{D(u^{n+1}, u^n)}{\|v^{n+1} - v^n\|} < \delta\}. \quad (24)$$

If $n_{ME}(\delta) \rightarrow \infty$ as $\delta \rightarrow 0$ and (17) holds, then $D(u^\dagger, u^{n_{ME}(\delta)}) \rightarrow 0$ as $\delta \rightarrow 0$ subsequentially.

Proof: Item (a) has already been proven by Proposition 2.

Due to (a) and (6), we get the first part of item (b) by virtue of

$$\|u^n\|_1 = \langle Av^n, u^n \rangle = \langle v^n, A^*u^n \rangle = \langle v^n, f^\delta \rangle.$$

Due to assumption (22), the feasible set for u^n contains the feasible set for u^{n+1} , hence (4) yields the second part of (b).

Note that

$$\begin{aligned} D(u^\dagger, u^{n+1}) - D(u^\dagger, u^n) &= \langle \xi^n - \xi^{n+1}, u^\dagger \rangle \\ &= -\langle A(v^{n+1} - v^n), u^\dagger \rangle \\ &= -\langle v^{n+1} - v^n, f^\delta \rangle + \langle v^{n+1} - v^n, f^\delta - f \rangle \\ &\leq -\langle v^{n+1} - v^n, f^\delta \rangle + \|v^{n+1} - v^n\| \delta \\ &= -(d_{ME}(n) - \delta) \|v^{n+1} - v^n\|. \end{aligned}$$

The first identity for $d_{ME}(n)$ in (c) is an immediate consequence of (b), while the second one follows in case of (22) from $\langle v^n, A^*u^{n+1} \rangle = \langle v^n, A^*u^n \rangle$ which can be rewritten as $\langle \xi^n, u^{n+1} - u^n \rangle = 0$.

For showing item (d), let $n_{AP}(\delta)$ be an a priori stopping rule satisfying (18), let $(\delta_k)_k$ be a sequence of noise levels tending to zero and denote by $n_{AP}^k = n_{AP}(\delta_k)$, $n_{ME}^k = n_{ME}(\delta_k)$ the stopping indices chosen by the a priori and the monotone error rule, respectively.

If there exists k_0 such that $n_{ME}^k > n_{AP}^k$ for all $k \geq k_0$, then by monotone decay of the error up to n_{ME}^k we have $D(u^\dagger, u^{n_{ME}^k}) \leq D(u^\dagger, u^{n_{AP}^k}) \rightarrow 0$ as $k \rightarrow \infty$ (actually one has strong convergence of the sequence $(u^{n_{AP}^k})_k$ to a solution). Otherwise there exists a subsequence $(k_l)_l$ such that for all $l \in \mathbb{N}$ we have $n_{ME}^{k_l} \leq n_{AP}^{k_l}$ and therefore, by (22), $\kappa_{n_{ME}^{k_l}} \leq \kappa_{n_{AP}^{k_l}}$, so the right hand limit in (18) together with the assumption $n_{ME}(\delta) \rightarrow \infty$ implies strong convergence of $(u^{n_{ME}^{k_l}})_l$ to a solution of the equation. \square

Besides the monotone error rule, which gives unconditional convergence, we also consider the discrepancy principle

$$n_{DP}(\delta) = \min\{n \in \mathbb{N} : \|A^*u^n - f^\delta\| \leq \tau\delta\} \quad (25)$$

with some fixed $\tau > 1$, for which, as usual (cf., e.g., conditions (2.13), (2.14) in [8]) certain assumptions on the discretization have to be made to guarantee well-definition and convergence. We assume existence of constants C_1, C_2 such that for all $n \in \mathbb{N}$

$$\kappa_n \gamma_n \leq C_1 \quad (26)$$

$$\kappa_n \hat{\gamma}_{n-1} \leq C_2 \quad (27)$$

where

$$\gamma_n = \sup_{\substack{u^1, u^2 \in E_n \\ D^{\text{sym}}(u^1, u^2) \neq 0}} \frac{\|(\text{id} - P_n)A^*(u^1 - u^2)\|}{D^{\text{sym}}(u^1, u^2)}, \quad \hat{\gamma}_n = \|(\text{id} - P_n)A^*(u^{\dagger, n} - u^\dagger)\| \quad (28)$$

with $u^{\dagger, n}$ as in (16), P_n as in (3) and E_n as in (9).

Theorem 3 Let the assumptions of Proposition 1 be satisfied, assume that (1) is solvable and that the noisy data f^δ satisfy (2). Additionally, assume that condition (26) with $\tau > 2C_1 + 1$ holds and that $\hat{\gamma}_n \rightarrow 0$ as $n \rightarrow \infty$. Then $n_{DP}(\delta)$ according to the discrepancy principle (25) is well-defined. If additionally (27) holds and the sequence $(\delta_m)_m$ in $(0, +\infty)$ converges to zero, then there exists a subsequence $(\delta_l)_l$ such that

$$u^l \xrightarrow{*} u^\dagger \text{ as } l \rightarrow \infty \text{ in } \ell^1, \quad (29)$$

with $u^l := u^{n_{DP}(\delta_l)}$ and u^\dagger a solution of (1).

Proof: Using (5), (14), (26), (28) we get

$$\begin{aligned} \|A^*u^n - f^\delta\| &= \left\| (\text{id} - P_n) \left(A^*(u^n - u^{\dagger,n}) + A^*(u^{\dagger,n} - u^\dagger) + (f - f^\delta) \right) \right\| \\ &\leq (2C_1 + 1)\delta + \hat{\gamma}_n, \end{aligned} \quad (30)$$

where $\hat{\gamma}_n$ tends to zero as $n \rightarrow \infty$, hence the right hand side is smaller than $\tau\delta$ for sufficiently large n . Consequently, $n_{DP}(\delta)$ is well-defined.

On the other hand, (30) together with minimality in (25) yields

$$(\tau - 2C_1 - 1)\delta < \hat{\gamma}_{n_{DP}(\delta)-1} \quad (31)$$

hence by (13) and (27) we have

$$\|u^{n_{DP}(\delta)}\|_1 \leq \delta \kappa_{n_{DP}(\delta)} + \|u^\dagger\|_1 \leq \frac{C_2}{\tau - 2C_1 - 1} + \|u^\dagger\|_1$$

which yields uniform boundedness of $(\|u^{n_{DP}(\delta_m)}\|_1)_m$, hence, as in the proof of Theorem 1, weak* subsequential convergence. \square

5. Convergence rates under a source condition

We assume throughout this section that A and consequently, A^* is injective and that the following source condition is satisfied:

Assumption 1 There exists a source element $v^\dagger \in H$ such that $\|Av^\dagger\|_\infty \leq 1$ and $(Av^\dagger)_i = \text{sgn}(u_i^\dagger)$ whenever $u_i^\dagger \neq 0$.

Note that $Av^\dagger \in c_0$, hence there are only finitely many i for which $(Av^\dagger)_i \in \{-1, 1\}$ and, consequently, only finitely many i with $u_i^\dagger \neq 0$. The latter means that the solution u^\dagger has to be sparse.

5.1. The structure of the source element

We first see that v^\dagger can be assumed, without loss of generality, to satisfy $|(Av^\dagger)_i| = 1$ if and only if $u_i^\dagger \neq 0$.

Lemma 2 There exists a $v^\dagger \in H$ with $\|Av^\dagger\|_\infty \leq 1$, $(Av^\dagger)_i = \text{sgn}(u_i^\dagger)$ whenever $u_i^\dagger \neq 0$ and $|(Av^\dagger)_i| < 1$ whenever $u_i^\dagger = 0$.

Proof: First, denote

$$I = \{i \in \mathbf{N} \mid (Av^\dagger)_i \in \{-1, 1\}\} \quad (32)$$

which is a finite set by $Av^\dagger \in c_0$, and

$$H_I = \text{span}\{A^*e_i \mid i \in I\} \quad (33)$$

where $(e_i)_i$ is the canonical basis in ℓ^1 . By injectivity of A^* , $\{A^*e_i \mid i \in I\}$ are linearly independent. Thus, denoting

$$A_I : H \rightarrow \mathbf{R}^I, \quad (A_I v)_i = (Av)_i \text{ for } i \in I, \quad (34)$$

we see that the mapping $A_I A_I^* : \mathbf{R}^I \rightarrow \mathbf{R}^I$ is (continuously) invertible. Thus, the problem of finding a $v_I \in H_I$ such that $\langle v_I, A^*e_i \rangle = \bar{v}_i$ for $i \in I$ and given $\bar{v} \in \mathbf{R}^I$, is uniquely solvable with $\|v_I\|_H \leq C \|\bar{v}\|_\infty$ for some $C > 0$ independent of \bar{v} . In particular, choosing $I_0 = \{i \in I \mid u_i^\dagger = 0\}$ and, for $\varepsilon > 0$,

$$\bar{v}_i = \begin{cases} -\varepsilon \operatorname{sgn}(Av^\dagger)_i & \text{if } i \in I_0, \\ 0 & \text{if } i \in I \setminus I_0, \end{cases}$$

we have, for the corresponding v_I , that $\|Av_I\|_\infty \leq C \|A\| \varepsilon$. Next, we know, again since $Av^\dagger \in c_0$, that

$$\rho = 1 - \max_{i \in \mathbf{N} \setminus I} |(Av^\dagger)_i| > 0. \quad (35)$$

Namely, for all $i \in \mathbf{N} \setminus I$, we have $|(Av^\dagger)_i| < 1$, and assuming $|(Av^\dagger)_{i_k}| \rightarrow 1$ for some subsequence contradicts $Av^\dagger \in c_0$. Choosing $0 < \varepsilon < \min(1, \frac{1}{2}(C\|A\|)^{-1}\rho)$ and letting $v^\ddagger = v^\dagger + v_I$ yield

$$\begin{cases} (Av^\ddagger)_i = \operatorname{sgn}(u_i^\dagger) & \text{for } i \in I \setminus I_0, \\ |(Av^\ddagger)_i| = 1 - \varepsilon < 1 & \text{for } i \in I_0, \\ |(Av^\ddagger)_i| \leq |(Av^\dagger)_i| + \|Av_I\|_\infty \leq 1 - \rho + C\|A\|\varepsilon \leq 1 - \frac{\rho}{2} < 1 & \text{for } i \in \mathbf{N} \setminus I. \end{cases}$$

This, however, immediately implies that v^\ddagger possesses the stated properties. \square

In the following assume that v^\dagger is a source element which satisfies $|(Av^\dagger)_i| = 1$ if and only if $i \in \operatorname{supp} u^\dagger$. Thus, from now on

$$I = \operatorname{supp} u^\dagger. \quad (36)$$

We denote by

$$\varepsilon_{v^\dagger} = 1 - \max_{i \notin \operatorname{supp} u^\dagger} |(Av^\dagger)_i| \quad (37)$$

which is a positive number since $Av^\dagger \in c_0$ (see the argument after (35)).

In the sequel, we will also make use of the projections operators P_n as in (3), in particular, the pointwise convergence $\lim_{n \rightarrow \infty} P_n v = v$ for each $v \in H$. Then, for n large enough, u^\dagger is already a solution of (5) for exact data, i.e., $f^\delta = f$.

Lemma 3 There is an $n_0 \in \mathbf{N}$ such that for $n \geq n_0$, u^\dagger is the unique solution to (5) with $f^\delta = f$.

Furthermore, for $n \geq n_0$, there is a source element $v^{\dagger,n} \in H_n$ with $\|Av^{\dagger,n}\|_\infty \leq 1$, $|(Av^{\dagger,n})_i| = 1$ if and only if $i \in \operatorname{supp} u^\dagger$ and with

$$\varepsilon_{v^{\dagger,n}} \geq \frac{1}{2} \varepsilon_{v^\dagger}. \quad (38)$$

Proof: Let I, H_I, A_I be defined as in (36), (33), (34). As $\|P_n A^* e_i - A^* e_i\|_H \rightarrow 0$ as $n \rightarrow \infty$ for each $i \in I$ and as I is finite, we also have convergence $P_n A_I^* \rightarrow A_I^*$ as $n \rightarrow \infty$ in the strong operator norm. Consequently, there is an $n_0 \in \mathbf{N}$ and a $C > 0$ such that for all $n \geq n_0$ we have that $A_I P_n A_I^*$ is invertible with $\|(A_I P_n A_I^*)^{-1}\| \leq C$, where the latter norm is the ∞ -1-operator norm for linear mappings $\mathbf{R}^I \rightarrow \mathbf{R}^I$. Hence, similarly to the proof of Lemma 2, the problem of finding a solution to

$$v_I^n \in P_n H_I : \quad \langle v_I^n, P_n A^* e_i \rangle = \bar{v}_i^n \quad \text{for } i \in I$$

for $n \geq n_0$ given the coefficients \bar{v}_i^n for $i \in I$, is well-posed and we have $\|v_I^n\|_H \leq C \|A\| \|\bar{v}^n\|_\infty$ for all $\bar{v}^n \in \mathbf{R}^I$.

By choosing n_0 possibly larger, we can achieve, as v^\dagger is chosen according to Lemma 2, for ρ as in (35) that $\|v^\dagger - P_n v^\dagger\|_H \leq \rho / (2 \|A\| (1 + C \|A\|))$ for all $n \geq n_0$. Then, we have for each $v_I^n \in P_n H_I$ satisfying $\langle v_I^n, A^* e_i \rangle = \langle v^\dagger - P_n v^\dagger, A^* e_i \rangle$, $\forall i \in I$ that $\|v_I^n\|_H \leq C \|A\| \|A(v^\dagger - P_n v^\dagger)\|_\infty$. Thus, $v^{\dagger,n} = P_n v^\dagger + v_I^n \in H_n$ satisfies

$$\langle v^{\dagger,n}, A^* e_i \rangle = \langle v^\dagger, P_n A^* e_i \rangle + \langle v^\dagger - P_n v^\dagger, A^* e_i \rangle = \langle v^\dagger, A^* e_i \rangle \in \{-1, 1\} \quad \text{for } i \in I$$

and, for $i \notin I$, we have by (35)

$$\begin{aligned} |\langle v^{\dagger,n}, A^* e_i \rangle| &= |(Av^{\dagger,n})_i| \leq |(Av^\dagger)_i| + \|A(P_n v^\dagger - v^\dagger)\|_\infty + \|Av_I^n\|_\infty \\ &\leq 1 - \rho + \frac{\rho \|A\|}{2 \|A\| (1 + C \|A\|)} + \frac{\rho C \|A\|^2}{2 \|A\| (1 + C \|A\|)} = 1 - \frac{\rho}{2} < 1. \end{aligned}$$

Consequently, for $n \geq n_0$, $v^{\dagger,n}$ obeys $\|Av^{\dagger,n}\|_\infty \leq 1$ and

$$\langle u^\dagger, Av^{\dagger,n} \rangle = \sum_{i \in I} (Av^\dagger)_i u_i^\dagger = \sum_{i \in I} |u_i^\dagger| = \|u^\dagger\|_1,$$

meaning that u^\dagger is a solution to (5) with $f^\delta = f$.

Next, suppose that u^* is another solution of (5) with $f^\delta = f$, which implies that $\langle v^{\dagger,n}, A^* u^* \rangle = \langle v^{\dagger,n}, f \rangle$ and $\|u^*\|_1 = \|u^\dagger\|_1$. Consequently, by construction of $v^{\dagger,n}$,

$$\langle u^*, Av^{\dagger,n} \rangle = \langle v^{\dagger,n}, f \rangle = \|u^\dagger\|_1 = \|u^*\|_1.$$

As u^* has the representation $u^* = \|u^*\|_1 (\sum_{k \in I} \alpha_k \sigma_k e_k + \sum_{k \notin I} \alpha_k \sigma_k e_k)$ with $\alpha_k \geq 0$, $\sigma_k \in \{-1, 1\}$, and $\sum_k \alpha_k = 1$, we deduce

$$\begin{aligned} \langle u^*, Av^{\dagger,n} \rangle &= \|u^*\|_1 \langle v^{\dagger,n}, \sum_{k \in I} \alpha_k \sigma_k A^* e_k + \sum_{k \notin I} \alpha_k \sigma_k A^* e_k \rangle \\ &\leq \|u^*\|_1 \left(\sum_{k \in I} \alpha_k + \sum_{k \notin I} \alpha_k |\langle v^{\dagger,n}, A^* e_k \rangle| \right). \end{aligned}$$

As $|\langle v^{\dagger,n}, A^* e_k \rangle| < 1$ for each $k \notin I$, we conclude that $\alpha_k = 0$ for each $k \notin I$ as otherwise, we would get the contradiction $\|u^*\|_1 < \|u^*\|_1$ from $\sum_{k \in \mathbf{N}} \alpha_k = 1$. Thus, identifying \mathbf{R}^I with the subspace of elements in ℓ^1 with support contained in I , we have $P_n A^* u^* = P_n A_I^* u^*$. Since $P_n A_I^*$ is invertible (see above) and $P_n A_I^* u^\dagger = P_n f = P_n A_I^* u^*$, it follows that $u^* = u^\dagger$, establishing uniqueness.

Finally, $v^{\dagger,n}$ satisfies the stated properties by construction. The construction also yields $\varepsilon_{v^{\dagger,n}} \geq \frac{1}{2} \varepsilon_{v^\dagger}$. \square

5.2. The structure of the approximations u^n

Next, consider f^δ such that $\|f^\delta - f\|_H \leq \delta$. To analyze the structure of a solution u^n , we consider the set

$$K_n = \overline{\text{conv}}(\{\sigma P_n A^* e_i \mid i \in \mathbf{N}, \sigma \in \{-1, 1\}\}) \subset H_n.$$

Note that this set coincides with the closed unit ball associated with the dual of the norm $v \mapsto \|Av\|_\infty$ on H_n . This set has an interior whose size can be estimated by $\frac{1}{\kappa_n}$, as for $v, w \in H_n$ we have that $\|Aw\|_\infty \leq 1$ implies $\|w\|_{H_n} \leq \kappa_n$, hence

$$\kappa_n \|v\|_{H_n} = \sup_{\|w\|_{H_n} \leq \kappa_n} \langle v, w \rangle \leq 1 \Rightarrow \sup_{\|Aw\|_\infty \leq 1} \langle v, w \rangle \leq 1,$$

i.e.,

$$\mathcal{B}_{1/\kappa_n}(0) \subseteq K_n. \quad (39)$$

Furthermore, $(e_i)_i \rightharpoonup^* 0$ as $i \rightarrow \infty$, so by weak*-to-weak continuity, $A^* e_i \rightharpoonup 0$ in H and $P_n A^* e_i \rightarrow 0$ in H_n . By (39), there exists $i_0 \in \mathbf{N}$ such that for all $i \geq i_0$, $\sigma P_n A^* e_i$ is not an extremal point of K_n . Consequently, K_n has only finitely many extremal points, i.e., is a convex polyhedron which is obviously also symmetric around 0. Furthermore, as K_n has non-empty interior, the dual ball $K_n^* \subset H_n$ is also a symmetric convex polyhedron, i.e., possesses a finite extremal point set $K_{n,\text{ex}}^*$. We see that

$$K_n = \bigcap_{v \in K_{n,\text{ex}}^*} \{\langle v, \cdot \rangle \leq 1\}.$$

Clearly, the extremal points of K_n form a symmetric subset of the following set: $\{\sigma P_n A^* e_i \mid i \in \mathbf{N}, \sigma \in \{-1, 1\}\}$. These can be associated with the indices i resulting in

$$I_n = \{i \in \mathbf{N} \mid P_n A^* e_i \text{ and } -P_n A^* e_i \text{ are extremal points of } K_n\} \quad (40)$$

which is a finite set with at least n elements (otherwise, K_n would have empty interior). By construction, each $v \in K_{n,\text{ex}}^*$ obeys $|\langle v, P_n A^* e_i \rangle| \leq 1$ for each $i \in \mathbf{N}$.

We observe that $K_{n,\text{ex}}^*$ and I_n have a major influence on the structure of solutions.

Lemma 4 For n fixed and each $f^\delta \in H$, there is a sparsest solution u_{sparse}^* to (5) with $(u_{\text{sparse}}^*)_i \neq 0$ for m distinct elements i in I_n , $(u_{\text{sparse}}^*)_i = 0$ else and $m \leq n$. In particular, any other solution u^{**} obeys $\#\{i \in \mathbf{N} \mid u_i^{**} \neq 0\} \geq m$.

Proof: Pick an u^* satisfying (5). As for $P_n f^\delta = 0$, the statement is obviously true for $u_{\text{sparse}}^* = 0$, we may assume, without loss of generality, that $P_n f^\delta \neq 0$ and $\|u^*\|_1 > 0$. Now, there is a $v^* \in H_n$ with $\|Av^*\|_\infty \leq 1$ and $\langle v^*, P_n A^* e_i \rangle = \text{sgn}(u_i^*)$ if $u_i^* \neq 0$. Moreover, for $\sigma_i^* = \text{sgn}(u_i^*)$ where $u_i^* \neq 0$, we may write

$$\frac{1}{\|u^*\|_1} P_n f^\delta = \sum_{u_i^* \neq 0} \frac{|u_i^*|}{\|u^*\|_1} \sigma_i^* P_n A^* e_i \quad \Rightarrow \quad \langle v^*, \frac{1}{\|u^*\|_1} P_n f^\delta \rangle = 1,$$

meaning that $\|u^*\|_1^{-1}P_nf^\delta$ is a convex combination of elements in $\{\langle v^*, \cdot \rangle = 1\} \cap K_n$. Now, the extremal points of $\{\langle v^*, \cdot \rangle = 1\} \cap K_n$ have to be extremal points of K_n : Otherwise, there is an extremal point $w \in \{\langle v^*, \cdot \rangle = 1\} \cap K_n$ which has a representation $w = \alpha w_1 + (1 - \alpha)w_2$ for $\alpha \in]0, 1[$ and $w_1, w_2 \in K_n$, $w_1 \neq w_2$. By the extremal point property, not both w_1 and w_2 can be contained in $\{\langle v^*, \cdot \rangle = 1\} \cap K_n$. Thus, $\langle v^*, w_1 \rangle \neq 1$ or $\langle v^*, w_2 \rangle \neq 1$. As $\alpha \langle v^*, w_1 \rangle + (1 - \alpha) \langle v^*, w_2 \rangle = 1$, either $\langle v^*, w_1 \rangle > 1$ or $\langle v^*, w_2 \rangle > 1$. This is, however, a contradiction to $\|Av^*\|_\infty \leq 1$ as $\langle v^*, \bar{w} \rangle \leq 1$ for all $\bar{w} \in K_n$. Consequently, w has to be an extremal point of K_n .

By Carathéodory's theorem, we know that $\frac{1}{\|u^*\|_1}P_nf^\delta$ is a convex combination of at most n extremal points of $\{\langle v^*, \cdot \rangle = 1\} \cap K_n$, and hence, of at most n extremal points of K_n , which implies

$$P_nf^\delta = \|u^*\|_1 \sum_{k=1}^n \alpha_k \sigma_k P_n A^* e_{i_k}$$

for $\alpha_1, \dots, \alpha_n \geq 0$, $\sum_{k=1}^n \alpha_k = 1$, $\sigma_k \in \{-1, 1\}$ and each $i_k \in I_n$. Thus, the minimum

$$\min \left\{ m \in \mathbf{N} \mid \exists i_1, \dots, i_m \in I_n, \alpha \in \mathbf{R}^m, \|\alpha\|_1 = \|u^*\|_1, \sum_{k=1}^m \alpha_k P_n A^* e_{i_k} = P_nf^\delta \right\}$$

exists and is finite. It is then clear that for an $\alpha \in \mathbf{R}^m$ associated with an optimal m we have $\alpha_k \neq 0$ for all $k = 1, \dots, m$. By construction, $u_{\text{sparse}}^* = \sum_{k=1}^m \alpha_k e_{i_k}$ with m admitting the above minimum and $\alpha \in \mathbf{R}^m$ according to the definition, is a sparsest solution. \square

Remark 3 From Lemma 3 it follows that there is a n_0 such that for $n \geq n_0$, u^\dagger is the sparsest solution with data f .

Lemma 5 Each u^* solution of (5) can be represented as a finite convex combination of solutions with minimal support in the following sense: A solution u^* has minimal support if for any other solution u^{**} with $\text{supp } u^{**} \subset \text{supp } u^*$ it follows $u^{**} = u^*$.

Proof: Obviously, the set of solutions S is a non-empty, convex and bounded subset of ℓ^1 . It is moreover contained in a finite-dimensional subspace of ℓ^1 . To see this, let u^* be a solution of (5) and $v^* \in H_n$ such that $\|Av^*\|_\infty \leq 1$ and $\langle Av^*, u^* \rangle = \|u^*\|_1$. Then, as A maps into c_0 , there is an i_0 such that for all $i \geq i_0$, $|Av_i^*| < 1$ and consequently, $u_i^* = 0$. For any other solution u^{**} we get

$$\langle Av^*, u^* \rangle = \langle v^*, P_n A^* u^* \rangle = \|u^*\|_1 = \|u^{**}\|_1 = \langle v^*, P_n A^* u^{**} \rangle = \langle Av^*, u^{**} \rangle.$$

Consequently, $u_i^{**} = 0$ for all $i \geq i_0$. Thus, S is contained in a finite-dimensional subspace of ℓ^1 .

Being a non-empty, convex and compact subset of a finite-dimensional space, each element in S can be represented by a finite convex combination of its extremal points. Let us verify that the extremal points satisfy the stated minimality property. For that purpose, let u^* be an extremal point of S with $u^* = \sum_{k=1}^{\kappa} u_k^* e_{i_k}$ for i_1, \dots, i_κ distinct and

each $u_k^* \neq 0$. Now, either the collection $\{P_n A^* e_{i_1}, \dots, P_n A^* e_{i_\kappa}\}$ is linearly independent or not. However, the case that these vectors are linearly dependent can be excluded as follows. Choose a $u = \sum_{k=1}^\kappa u_k e_{i_k} \neq 0$ such that $P_n A^* u = 0$. Then, for $\varepsilon > 0$ small enough we can achieve that $u^\varepsilon = u^* + \varepsilon u$ as well as $u^* - \varepsilon u$ are still solutions: Indeed, $P_n A^* u^{\pm\varepsilon} = P_n A^* u^*$ is satisfied for each $\varepsilon > 0$. Additionally, for $\text{sgn}(u_k^{\pm\varepsilon}) = \text{sgn}(u_k^*)$ for each k (which can be achieved for ε small enough) we have $\langle Av^*, u^{\pm\varepsilon} \rangle = \|u^{\pm\varepsilon}\|_1$, meaning that $u^{\pm\varepsilon}$ is a solution. However, $u^* = \frac{1}{2}u^\varepsilon + \frac{1}{2}u^{-\varepsilon}$ and $u^\varepsilon \neq u^{-\varepsilon}$, so u^* cannot be an extremal point. Consequently, $\{P_n A^* e_{i_1}, \dots, P_n A^* e_{i_\kappa}\}$ are linearly independent. Thus, if u^{**} is a solution with $\text{supp } u^{**} \subset \text{supp } u^*$, we have the representation $u^{**} = \sum_{k=1}^\kappa u_k^{**} e_{i_k}$. However, $P_n A^* u^* = P_n A^* u^{**}$ and by injectivity of $(u_1, \dots, u_\kappa) \mapsto P_n A^* \sum_{k=1}^\kappa u_k e_{i_k}$, $u_k^* = u_k^{**}$ for all $k = 1, \dots, \kappa$, i.e., $u^* = u^{**}$. \square

5.3. Error estimates

For proving an $O(\delta)$ convergence rate, we will assume that n is that large to ensure u^\dagger is the sparsest solution to (5) for the data f . This means in particular that $\text{supp } u^\dagger \subset I_n$ (cf. (40)) with $\{P_n A^* e_i \mid i \in \text{supp } u^\dagger\}$ consisting of linearly independent vectors. Let $f^\delta \in H$ be such that (2) holds for $\delta > 0$. Denote by u^n a solution of (5) which has to be sparse as for the corresponding source element $v^n \in H_n$, we have $Av^n \in c_0$. Lemma 4 states, however, that without loss of generality, $\text{supp } u^n$ possesses at most n elements.

Theorem 4 There exists a $C > 0$ and an n_0 such that for $n \geq n_0$ and $f^\delta \in H$ with $\|f^\delta - f\|_1 \leq \delta$, for any solution u^n of (5) with data f^δ it holds that

$$\|u^n - u^\dagger\|_1 \leq C\delta\kappa_n.$$

Proof: Choosing n_0 according to Lemma 3 and, for $n \geq n_0$, denoting by $v^{\dagger,n} \in H_n$ the source element associated with the solution u^\dagger with data f according to Lemma 3, and by u^n a sparse solution of (5) according to Lemma 4 with data f^δ with source element $v^n \in H_n$, we have, according to Proposition 3 for the Bregman distance D associated with the subgradient element $Av^{\dagger,n}$ and D^{sym} the symmetric Bregman distance associated with the subgradient elements $v^{\dagger,n}$ and v^n , respectively, that

$$D(u^n, u^\dagger) \leq D^{\text{sym}}(u^n, u^\dagger) \leq 2\kappa_n \|f^\delta - f\|_H \leq 2\delta\kappa_n.$$

On the other hand, by definition and with the operator P defined by $(Pu^n)_i = u_i^n$ if $i \notin \text{supp } u^\dagger$ and 0 otherwise, we have

$$D(u^n, u^\dagger) \geq \sum_{i \notin \text{supp } u^\dagger} (1 - |(Av^{\dagger,n})_i|) |u_i^n| \geq \varepsilon_{v^{\dagger,n}} \|Pu^n\|_1 \geq \frac{\varepsilon_{v^{\dagger,n}}}{2} \|Pu^n\|_1.$$

In total, with $Pu^\dagger = 0$, one concludes

$$\|P(u^n - u^\dagger)\|_1 = \|Pu^n\|_1 \leq \frac{4\delta\kappa_n}{\varepsilon_{v^{\dagger,n}}}.$$

Furthermore, for $Q = \text{id} - P$ we see by $Qu^\dagger = u^\dagger$ that

$$\begin{aligned} \|P_n A^* Q(u^n - u^\dagger)\|_H &= \|P_n(f^\delta - f) - P_n A^* P u^n\|_H \\ &\leq \delta + \|A\| \frac{4\delta\kappa_n}{\varepsilon_{v^\dagger}} \leq \frac{\varepsilon_{v^\dagger} \kappa_n^{-1} + 4\|A\|}{\varepsilon_{v^\dagger}} \delta\kappa_n \leq \frac{(\varepsilon_{v^\dagger} + 4)\|A\|}{\varepsilon_{v^\dagger}} \delta\kappa_n \end{aligned}$$

where $\kappa_n^{-1} \leq \|A\|$ follows from the definition of κ_n in (11). Now, for I as in (36), A_I as in (34), we can choose $C_I > 0$ such that $\|(A_I P_n A_I^*)^{-1}\| \leq C_I$ for all $n \geq n_0$. Consequently,

$$\begin{aligned} \|Q(u^n - u^\dagger)\|_1 &= \|(A_I P_n A_I^*)^{-1} A_I P_n A^* Q(u^n - u^\dagger)\|_1 \\ &\leq \frac{C_I(\varepsilon_{v^\dagger} + 4)\|A\|^2 \delta\kappa_n}{\varepsilon_{v^\dagger}}. \end{aligned}$$

In total, we have

$$\|u^n - u^\dagger\|_1 \leq \frac{4 + C_I(\varepsilon_{v^\dagger} + 4)\|A\|^2}{\varepsilon_{v^\dagger}} \delta\kappa_n = C\delta\kappa_n.$$

which is the desired statement. \square

With this result, the choice $n = n_0$ gives an $O(\delta)$ estimate of the error $\|u^n - u^\dagger\|_1$. However, n_0 is not known a priori. We now show that under certain assumptions it can be replaced by $n = n_{DP}(\delta)$ according to the discrepancy principle, again for general solutions of (5) and without needing $v^{\dagger,n}$ from Lemma 3, but just relying on the source condition Assumption 1 and the specially constructed source element $v^\dagger = v^\dagger_\dagger$ from Lemma 2. Note however, that this will — besides requiring additional assumptions such as (26) — typically also not lead to the ideal rate $O(\delta)$.

Theorem 5 Let (26) hold. Then there exists $C > 0$ such that for $f^\delta \in H$ satisfying (2) and any solution $u_{n_{DP}(\delta)}$ of (5) with $n = n_{DP}(\delta)$ according to (25), it holds that

$$\|u^{n_{DP}(\delta)} - u^\dagger\|_1 \leq C\delta\kappa_{n_{DP}(\delta)}. \quad (41)$$

If additionally, for some $C_3 > 0$ and all $n \in \mathbb{N}$,

$$\kappa_n \|(\text{id} - P_{n-1})A^*\| \leq C_3 \quad (42)$$

and, for some index function Ψ , (i.e., a strictly monotone function satisfying $\Psi \rightarrow 0$ as $t \rightarrow 0$)

$$\|A^*(u^{\dagger,n} - u^\dagger)\| \leq \Psi\left(\frac{1}{\kappa_{n+1}}\right) \quad (43)$$

holds, then

$$\|u^{n_{DP}(\delta)} - u^\dagger\|_1 = O\left(\frac{\delta}{\Phi^{-1}(\frac{\delta}{C})}\right), \quad (44)$$

where $\Phi(\lambda) = \lambda\Psi(\lambda)$ and $\tilde{C} > 0$ is a constant independent of δ .

Proof: With $n = n_{DP}(\delta)$ and using (13) we get that

$$\begin{aligned} D(u^n, u^\dagger) &= \|u^n\|_1 - \langle Av^\dagger, u^n \rangle \\ &= \|u^n\|_1 - \langle Av^\dagger, u^\dagger \rangle + \langle v^\dagger, f - f^\delta \rangle - \langle v^\dagger, A^*u^n - f^\delta \rangle \\ &\leq \left(\kappa_n + (\tau + 1) \|v^\dagger\| \right) \delta. \end{aligned}$$

Similarly to the proof of Theorem 4 above but a bit simpler (since we do not need the source elements $v^{\dagger,n}$ here) we get

$$D(u^n, u^\dagger) \geq \sum_{i \notin \text{supp } u^\dagger} (1 - |(Av^\dagger)_i|) |u^n| \geq \varepsilon_{v^\dagger} \|Pu^n\|_1,$$

hence

$$\|Pu^n\|_1 = \|P(u^n - u^\dagger)\|_1 \leq \frac{\kappa_n + (\tau + 1) \|v^\dagger\|}{\varepsilon_{v^\dagger}} \delta. \quad (45)$$

On the other hand,

$$\begin{aligned} \|A^*(\text{id} - P)(u^n - u^\dagger)\| &\leq \|A^*(u^n - u^\dagger)\| + \|A\| \|P(u^n - u^\dagger)\|_1 \\ &= \|A^*u^n - f\| + \|A\| \|P(u^n - u^\dagger)\|_1 \leq \left(\tau + 1 + \|A\| \frac{\kappa_n + (\tau + 1) \|v^\dagger\|}{\varepsilon_{v^\dagger}} \right) \delta, \end{aligned}$$

hence by boundedness of $(A^*(\text{id} - P))^\dagger := ((A^*(\text{id} - P))^* A^*(\text{id} - P))^{-1} (A^*(\text{id} - P))^*$ (see the proof of Lemma 2) and the fact that $(A^*(\text{id} - P))^\dagger A^*(\text{id} - P) = (\text{id} - P)$ we get

$$\|(\text{id} - P)(u^n - u^\dagger)\|_1 \leq C \left(\tau + 1 + \|A\| \frac{\kappa_n + (\tau + 1) \|v^\dagger\|}{\varepsilon_{v^\dagger}} \right) \delta. \quad (46)$$

Combining (45), (46) yields (41).

To obtain a convergence rate with respect to δ it is essential to estimate $\hat{\gamma}_n$ from above and below. For the former purpose, we proceed as above, but this time for the noise free discrete approximation and using the infinite dimensional source element v^\dagger . Namely, using the fact that by minimality $\|u^{\dagger,n}\|_1 \leq \|u^\dagger\|_1$,

$$\begin{aligned} D(u^{\dagger,n}, u^\dagger) &= \|u^{\dagger,n}\|_1 - \langle Av^\dagger, u^{\dagger,n} \rangle \leq \langle v^\dagger, A^*(u^\dagger - u^{\dagger,n}) \rangle, \\ D(u^{\dagger,n}, u^\dagger) &\geq \sum_{i \notin \text{supp } u^\dagger} (1 - |(Av^\dagger)_i|) |u^{\dagger,n}| \geq \varepsilon_{v^\dagger} \|Pu^{\dagger,n}\|_1, \end{aligned}$$

one has

$$\|Pu^{\dagger,n}\|_1 = \|P(u^{\dagger,n} - u^\dagger)\|_1 \leq \frac{\|v^\dagger\|}{\varepsilon_{v^\dagger}} \|A^*(u^{\dagger,n} - u^\dagger)\|$$

as well as

$$\begin{aligned} \|(\text{id} - P)(u^{\dagger,n} - u^\dagger)\|_1 &\leq C \|A^*(\text{id} - P)(u^{\dagger,n} - u^\dagger)\| \\ &\leq C \|A^*(u^{\dagger,n} - u^\dagger)\| + C \|A\| \|P(u^{\dagger,n} - u^\dagger)\|_1 \leq C \left(1 + \frac{\|A\| \|v^\dagger\|}{\varepsilon_{v^\dagger}} \right) \|A^*(u^{\dagger,n} - u^\dagger)\|. \end{aligned}$$

Altogether, one obtains

$$\|u^{\dagger,n} - u^{\dagger}\|_1 \leq \left(C + (1 + C\|A\|) \frac{\|v^{\dagger}\|}{\varepsilon_{v^{\dagger}}} \right) \|A^*(u^{\dagger,n} - u^{\dagger})\|.$$

Inserting this into the definition of $\hat{\gamma}_n$ (28) and using (42), (43) yields

$$\begin{aligned} \hat{\gamma}_n &= \|(\text{id} - P_n)A^*(u^{\dagger,n} - u^{\dagger})\| \leq \|(\text{id} - P_n)A^*\| \|u^{\dagger,n} - u^{\dagger}\| \\ &\leq \frac{C_3}{\kappa_{n+1}} \left(C + (1 + C\|A\|) \frac{\|v^{\dagger}\|}{\varepsilon_{v^{\dagger}}} \right) \|A^*(u^{\dagger,n} - u^{\dagger})\| \leq \bar{C} \frac{\Psi(\frac{1}{\kappa_{n+1}})}{\kappa_{n+1}} \end{aligned}$$

Thus from (31) we conclude

$$\delta \leq \tilde{C} \Phi \left(\frac{1}{\kappa_{n_{DP}(\delta)}} \right),$$

i.e.,

$$\frac{1}{\Phi^{-1} \left(\frac{\delta}{\tilde{C}} \right)} \geq \kappa_{n_{DP}(\delta)}.$$

Inserting this into (41) yields (44). \square

6. Particularities of the method

The aim of this section is to deeper understand the effect of the least error method as a discretization method for relevant bases.

(i) The case of the singular basis

Let, for \mathcal{H} a Hilbert space, the linear operator $\mathcal{A} : H \rightarrow \mathcal{H}$ be compact and let $(\sigma_n, \hat{v}^n, \hat{u}^n)$ be a singular basis of the compact operator \mathcal{A} . Here, $(\sigma_n)_n$ stands for the non-increasingly ordered sequences of positive singular values converging to zero as $n \rightarrow \infty$ and $(\hat{v}^n)_n, (\hat{u}^n)_n$ are orthonormal systems in H and \mathcal{H} , respectively. Then,

$$\mathcal{A}^*u = \sum_i \sigma_i \langle u, \hat{u}^i \rangle \hat{v}^i$$

and with the basis operator $T : \mathcal{H} \rightarrow c_0$, $(Tu)_i = \langle u, \hat{u}^i \rangle$, the least error framework for the solution of $\mathcal{A}^*u = f^\delta$ may be applied to $A = T\mathcal{A}$. With the choice $H_n = \text{span}(\hat{v}^1, \dots, \hat{v}^n)$, this results in

$$u^n \in \text{argmin}\{ \|Tu\|_1 : \forall i = 1, 2, \dots, n : \sigma_i \langle u, \hat{u}^i \rangle = \langle \hat{v}^i, f^\delta \rangle \}. \quad (47)$$

(ii) The case of the canonical basis

Consider the canonical basis $(e_n)_n$ in $H = \ell^2$ and $H_n = \text{span}(e_1, \dots, e_n)$. Thus, one can re-formulate (4) as

$$u^n \in \text{argmin}\{ \|u\|_1 : \forall i = 1, 2, \dots, n : (A^*u)_i = f_i^\delta \}. \quad (48)$$

If one considers the denoising problem, then the operator A^* in this case is just the embedding operator from ℓ^1 into ℓ^2 and

$$u^n \in \operatorname{argmin} \left\{ \sum_{i>n} |u_i| : \forall i = 1, 2, \dots, n : u_i = f_i^\delta \right\}, \quad (49)$$

where the first n components of the regularized solution u^n coincide with the first n components of the noisy data. Since the minimizer of the above problem is attained when $u_i = 0$, for all $i > n$, one obtains

$$u^n = (f_1^\delta, f_2^\delta, \dots, f_n^\delta, 0, 0, \dots).$$

7. Conclusions and Remarks

In this paper we have provided a stability and convergence analysis for the least error method with ℓ^1 as a preimage space. We have proven convergence rates under a source condition, even with respect to the norm topology, and shown that the method indeed leads to sparse approximations. The analysis includes detailed investigations on the source elements, which are crucial for stability estimates leading to ideal $O(\delta)$ convergence rates.

Future research will be concerned with an efficient implementation of the method as well as numerical tests. Moreover we are working on an extension of the approach to a sparsity enhancing method in a function space setting with spaces of Radon measures in place of ℓ^1 .

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